MAT332H1: Introduction to Graph Theory

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Notes are based on the lectures of `MAT332H1` *at the University of Toronto Fall 2024 taught by Professor Hunter Spink. As there are no lecture notes or slides for the course, these notes are based on the blackboard scribbles during the lecture. There will be typos, and there will be proofs that seem a bit 'hand wavy'. Please feel free to let me know any typos or mistakes by email* `hepaul@cs.toronto.edu`.

Contents

1 Introduction to Graph Theory

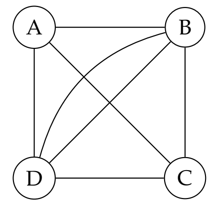


Illustration of the Königsberg Bridge Problem as a (multi)-graph.

The Königsberg Bridge Problem The city occupied two islands plus areas on both banks. These regions were linked by seven bridges. Euler asked himself in 1736 whether he can walk across all seven bridges *exactly* once? The answer is no!

The first question we should ask ourselves is, given a string of letters, how can we verify whether it is a valid walk?

- We can store the number of edges between each vertex.
- When you enter a vertex, you must leave it in the next step! Therefore, we can say there is a *natural pairing* of all the edges used (parity problem). This holds for any vertex except for the first and the last.

To prove that there is no such solution, we do this by contradiction. First, we prove a claim.

Claim: If a vertex X is *not at the start or end*, then the number of edges incident to X is even.

Proof. Let X be some arbitrary vertex, and assume that X is not at the start or the end. Then, given a walk of length k as a sequence of strings, we note that there are not two consecutive X's in the walk. Namely, for $X \notin \{Y_1, \ldots, Y_k\}$ that appears before an X, and $X \notin \{Z_1, \ldots Z_k\}$ that appear after, the walk takes the form of

$$Y_1 X Z_1 \ldots Y_2 X Z_2 \ldots Y_k X Z_k \tag{1}$$

Then, it is clear we can see that there are $2k$ edges incident to X. □

Now we can proof the original claim, that there is no such solution to the question Euler asked.

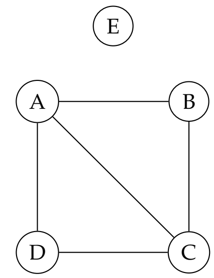
Proof. For the sake of contradiction, assume that there was such a walk. Then, in the original Königsberg problem as illustrated, all verticies have an odd number of edges, by the claim we know that **all would have** to start/end the sequence of walk, which is a contradiction to the original claim. □

1.1 Basic Definitions

Graph (Informal)
A graph G is consists of a finite set V called the verticies and subset E of the unordered pairs of verticies in V.

Note that this definition forbits directed graphs, multiple edges (set restriction), and self loops.

Graph
A graph G is a pair $G = (V, E)$ where V is a finite set, and $E \subseteq \left\{\{v, w\} : v, w \in V \wedge v \neq w\right\}$. It is common to denote E as $V^{(2)}$, which represents all possible edges in a graph with vertex set V. The (2) denotes a binary relation, we values > 2 we would have hypergraphs.

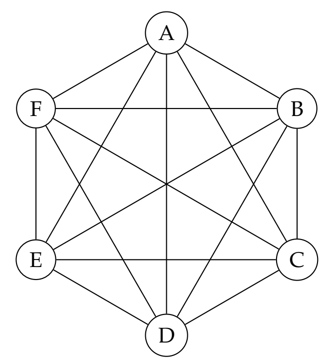


A random undirected graph. Here we have $V(G) = \left\{A, B, C, D, E\right\}$ and $E(G) = \left\{\{A, B\}, \{A, C\}, \{B, C\}, \{C, D\}, \{D, A\}\right\}$ but for simplicity $E(G) = \left\{AB, AC, BC, CD, DA\right\}$.

Vertex, Edge set
Given a graph G, we use $V(G)$ to denote the vertex set, $E(G)$ to denote the edge set when we need disambiguation.

Order
The order of a graph G is denote as the number of verticies $|G| := |V(G)|$

Size
The size of a graph G is denote as the number of edges $e(G) := |E(G)|$



A fully connected graph.

If $|G| = n$, then the possible values for $e(G)$ range from $0 \leq e(G) \leq \frac{n(n-1)}{2}$.

Neighborhood
The neighborhood of $v \in V$ is defined as

$$\Gamma(v) = N(v) = \left\{w \in V : vw \in E\right\} \tag{2}$$

if $\Gamma(v) = \varnothing$ we call it an **isolated vertex**.

Degree
For $v \in V$, the degree of v is $d_v = |\Gamma(v)|$.

Complete Graph
A complete graph K_n has vertex set $V = \{1, \ldots n\}$ and edge set $V^{(2)} = \{ij : 1 \leq j < j \leq n\}$ which is also equivalent to $\{ij : 1 \leq j, j, \leq n, i \neq j\}$

Empty Graph
An empty graph $\overline{K_n}$ has vertex set $V = \{1, \ldots n\}$ and edge set $E = \varnothing$.

Complement
Given a graph $G = (V, E)$, we define the complement of G as $\overline{G} = (V, V^2 \backslash E)$

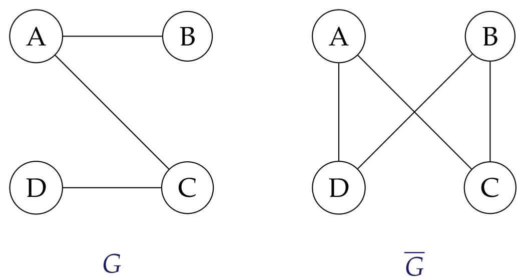


Figure 1. An example of a graph G and its complement $\overline{G}$.

Path Graph
A path graph P_n has the vertex set $V = \{1, \ldots n\}$ and edge set $E = \{i(i+1) : 1 \leq i \leq n-1\} \cup \{n1\}$

Figure 2. An example of a path graph P_4.

Cycle Graph
A path graph P_n has the vertex set $V = \{1, \ldots n\}$ and edge set $E = \{i(i+1) : 1 \leq i \leq n-1\} \cup \{n1\}$

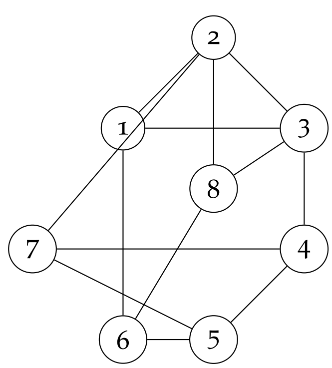


Figure 3. An example of a 3-regular graph with 8 vertices.

K-Regular Graph
A k-regular graph is a graph G where $d_V = k$, for all $v \in V$.

1.2 *The Handshaking Lemma*

Notice that we could not for example, draw a 3-Regular 9 Vertex Graph. This is restricted due to the handshaking lemma.

Handshaking Lemma
For a graph $G = (V, E)$, the sum of the degrees of all vertices in the graph is equal to twice the number of edges. We have that

$$2e(G) = \sum_{v \in V(G)} d_v \tag{3}$$

or

$$e(G) = \frac{1}{2} \sum_{v \in V(G)} d_v \tag{4}$$

which can be understood as the *parity constraint*.

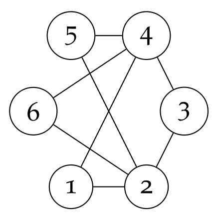


Figure 4. The sum of all degrees is 16, applying the handshaking lemma the number of edges is indeed 8.

This introduces two different proof methods. The first proof method is by double counting, where the idea is to count the number of pairs (v, e) with $v \in V$, $e \in E$ and $v \in e$ (meaning v is incident to e).

Proof. Let $G = (V, E)$, we count the number of pairs (v, e) with $v \in V$, $e \in E$ and $v \in e$. Indeed, for a fixed v, the number of edges incident to it $n_{v,e}$ is the number of pairs $(v, e) = d_v$. Therefore, the number of such pairs, denoted as N is

$$\begin{aligned} N &= \sum_{v \in V} n_{v,e} && (5) \\ &= \sum_{v \in V} d_v && (6) \\ &= \sum_{e \in E} n_{v,e} && (7) \\ &= \sum_{e \in E} 2 && (8) \\ &= 2e(G) && (9) \end{aligned}$$

Notice that $\sum_{e \in E} n_{v,e} = \sum_{e \in E} 2$ since we are summing over each edge, and by definition each edge will only be incident 2 vertices only. □

An intuitive way to understand the double counting proof is via the *incident matrix* (which is flipped around as to the adjacency matrix). For the graph illustrated by fig. 4, we have the following incident matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{10}$$

The double counting proof is asking "how many 1's are there in this matrix", notice that one way is add by row (sum of degree) but notice if we do this column wise it is just a sum over 2.

The second proof technique is via (strong) induction.

Proof. Let $G = (V, E)$ be a graph. We proof by strong induction over the number of edges $e(G)$.

Base Case: $e(G) = 0$. Then, we can take $G = \overline{K_n}$ where $\overline{K_n}$ is any empty graph. We have (by definition of the empty graph)

$$2e(\overline{K_n}) = 0 = \sum_{v \in V(\overline{K_n})} d_v = 0 \tag{11}$$

which completes the base case.

Step Case: As our inductive hypothesis, assume that for any graph H where $e(H) < e(G)$ that the result (handshaking lemma) holds. We want to show that $2e(G) = \sum_{v \in V(G)} d_v$. Pick an edge $e = xy$, then by our inductive hypothesis we know [1]

$$2e(G-e) = \sum_{v \in V(G-e)} d_{v,G-e} \tag{12}$$

This is equivalent to

$$2\bigg(e(G) - 1\bigg) = d_{x,G-e} + d_{y,G-e} + \sum_{v \in V \setminus \{x,y\}} d_{v,G-e} \tag{13}$$

Therefore,

$$\begin{aligned} 2e(G) &= 2\bigg(e(G) - 1\bigg) + 2 && (14) \\ &= d_{x,G-e} + d_{y,G-e} + \sum_{v \in V \setminus \{x,y\}} d_{v,G-e} + 2 && (15) \\ &= \sum_{v \in V(G)} d_{v,G} && (16) \end{aligned}$$

□

[1] We introduce this definition later but for now $G - e$ denotes the graph G with the edge e removed.

Similarly, we could've chosen to induct of the number of vertices, where the base case would be a graph with no vertices, and the step case to assume true on a graph $G - v$.

Removals
Let $G = (V, E)$ be a graph, we **remove an edge** $e \in E$ and define $G - e := (v, E \backslash \{e\})$.
Similarly, if $v \in V$, we **remove a vertex** $v \in V$ we define $G - v := (V \backslash \{v\}, E \backslash \{vw : vw \in E\})$

Subgraphs
A graph H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Induced Subgraphs
Let $U \subseteq V(G)$ be a subset of the vertex set of a graph G. The graph $H = G[U]$ is an **induced subgraph** of G if:

- $V(H) = U$, and
- $E(H) = \{vw \in E(G) : v, w \in U\}$.

This means that H is formed by taking all vertices in U and all edges between them that appear in G.

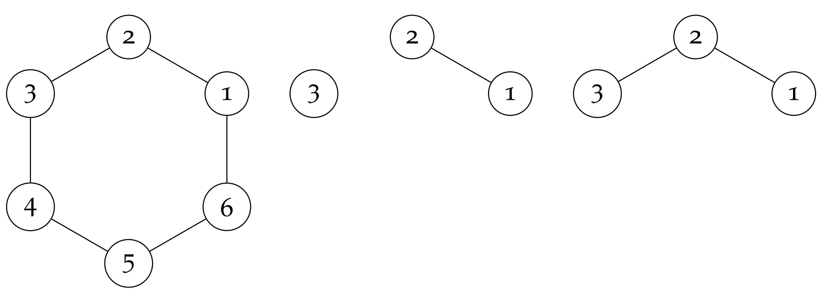


Figure 5. Left: The original graph G, Middle: H_1 a subgraph but *not* induced, Right H_2, an induced subgraph, where G is essentially restricted to vertices $1, 2, 3$. Therefore, if you include all the vertices you get the original graph back.

1.3 *Isomorphism*

The goal of this section is to ask the question whether a graph G contains a copy of the graph H. This means that we will need to talk about graphs without labels!

Isomorphism
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. An isomorphism $f : G_1 \to G_2$ is a bijection $f : V(G_1) \to V(G_2)$ such that $wv \in E(G_1) \iff f(v)f(w) \in E(G_2)$. We say that G_1 is **isomorphic** to G_2, written $G_1 \cong G_2$, if there is an isomorphism from G_1 to G_2.

Notice that $\cong$ is an equivalence relation for graphs, namely:

1. **Reflexivivty**: $G \cong G$.
2. **Symmetric**: If $f : V(G_1) \to V(G_2)$ is an isomorphism from G_1 to G_2, then f^{-1} is an isomorphism from G_2 to G_1 by definition of isomorphism.
3. **Transitivity**: $G_1 \cong G_2$ and $G_2 \cong G_3$, then $G_1 \cong G_3$.

Consider the graphs in fig. 6, indeed, we can find a bijection $f(x) = x + 6$, $f^{-1}(y) = y - 6$ to find the isomorphism between G_1 and G_2.

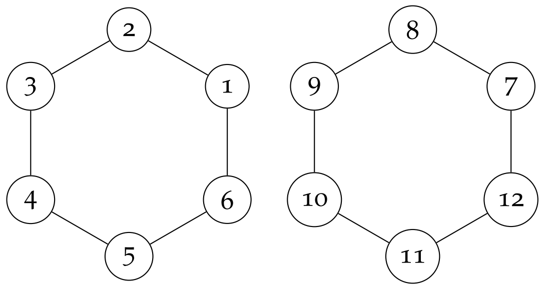


Figure 6. Left: The original graph G_1, Right: another graph G_2, we can find a mapping that maps the vertices from G_1 to G_2, in fact, the mapping is not unique!

There are two techniques to show that this is an isomorphism. First we assume that there exists a bijection on vertices.

1. If you have $vw \in E(G_1)$ then $f(v)f(w) \in E(G_2)$ *and* if $xy \in E(G_2)$ then $f^-1(x)f^-1(y) \in E(G_1)$
2. If $vw \in E(G_1)$ then $f(v)f(w) \in E(G_2)$ *and* if $vw \notin E(G_1)$ then $f(v)f(w) \notin E(G_2)$.

We say that a graph G contains a **copy** of a graph H if H is isomorphic to a subgraph of G.

We say a graph G contains an **induced copy** of a graph H if H is an induced subgraph of G.

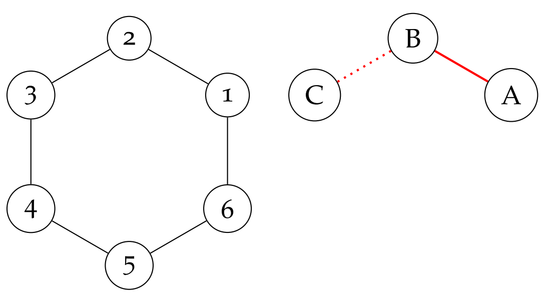


Figure 7. Left: Graph G_1 with a highlighted subgraph isomorphic to graph H, Right: Graph G_2 also containing a copy of H. We say G_1 contains a **copy** of H since H is isomorphic to a subgraph of G_1. For example, we could map ABC to $16X$ where X is any other vertex.

A **clique** is a subset $U \subseteq V(G)$ such that $G[U]$ is complete.

Can think of this as a collection of friends on Facebook.

An **independent set** is a subset $U \subseteq V(G)$ such that $G[U]$ is empty.

Similarly, we can think of this as strangers which haven't met.

Now suppose we have a graph G and its complement $\overline{G}$. A clique in G turns into an independent set, an independent set in G turns into a clique.

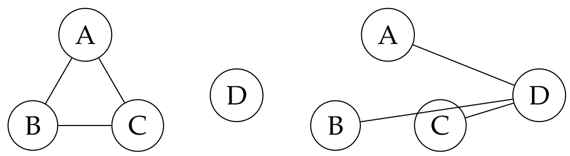


Figure 8. Graph G (left) and its complement $\overline{G}$ (right). The triangle (clique) in G becomes an independent set in $\overline{G}$, and the isolated vertex in G becomes a clique with the other vertices in $\overline{G}$.

We can split up graphs into connected components, this allows us to reduce any graph problems into connected graphs!.

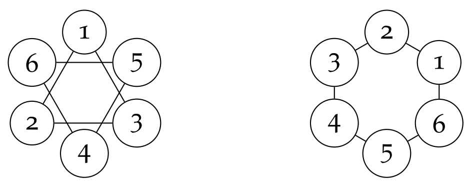


Figure 9. What makes these graphs different? We can talk about walks! For example a walk 123456 is valid on the right but not possible on the left.

There is a **walk** from v to w where a walk from v to w is a sequence of vertices $X_1, X_2, \dots, X_k$ where $X_1 = v, X_k = w$ *and* $x_i x_{i+1} \in E(G)$ for $i \in \{1, \dots k-1\}$.

Linkage $\sim$ is an equivalence relation of $V(G)$.

1. **Reflexivity** $v \sim v$ indeed, the walk is v is satisfied.
2. **Symmetry** $v \sim w \implies w \sim v$.
3. **Transitivity** $x \sim y$ and $y \sim z \implies x \sim z$.

Corllary: For all vertices of G, decompose independent set $V(G) = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$, if $x, y \in V_i$ then $x \sim y$. If $x \in V_i$ and $y \in V_j$ with $i \neq j$ then $x \nsim y$.

Theorem: Every graph G has a unique decomposition $G = G_1 \sqcup G_2 \sqcup \cdots \sqcup G_k$ where each G_i is *connected* and *non-empty*.

Note that a graph is connected if every pair of vertices is linked. The following is the existence proof:

Proof. Let $G = (V, E)$ be arbitrary graphs, we show that the vertex and edge are subset First, note $V(G) = V_1 \sqcup \cdots \sqcup V_k$ (by definition).

Then, if $xy \in E(G[V_1] \sqcup \cdots \sqcup G[V_k])$. Then, $xy \in E(G[V_1])$ for some i for $xy \in E(G)$.

Suppose $V = V_1 \sqcup \cdots \sqcup V_k$ with link equivalences. Conversely, if $xy \in E(G)$, then if x, y are in the same part V_i then $xy \in E(G[V_i])$ by definition of $G[V_i]$, so $xy \in E(G[V_1] \sqcup \cdots \sqcup G[V_k]))$.

If $x \in V_i, y \in V_j$ and $i \neq j$, note that $xy \in E(G) \implies xy$ are linked, which means they must come from the same part, so $i = j$. Contradiction!. □

Proof for uniqueness.

Proof. Left as an exercise. □

Define $\Delta(G) = \max_{v \in V(G)} d_v$ which is the maximum degree of any vertex v in a graph G.

Theorem: $\Delta(G) \leq 2$ if and only if G is a disjoint union of paths and cycles.

Proof. $\implies$: Easy! Paths and cycles violate this.

$\impliedby$: $G = G_1 \sqcup G_2 \sqcup \cdots \sqcup G_k$. By the connected component decomposition, it suffices to show the theorem is true connected graphs. We show this by considering the longest path it G, call the path $X_1 \ldots X_k$. Let $v \in V(G) \backslash X_1 \ldots X_k$, $vX_1 \notin E(G), \forall 2 \leq i \leq k-1$ because $d_{vX_i} \geq 3$. For $i \in \{1, k\}$, $vX_i, vX_k \notin E(G)$ because then $vX_1 \ldots X_k$ or $X_1 \ldots X_k v$ would be a bigger path, which implies $X_1 \ldots X_k$ is connected component of G, which also implies $X_1 \ldots X_k$ are all vertices of G. □

Theorem: If $x \sim y$, then there is a path from x to y.

Proof. Consider the *smallest walk* from x to y. If this walk is already a path then the theorem holds. If it's not a path, consider the walk $v_1 \ldots v_i \ldots v_j \ldots v_k$ where $v_1 = x$ and $v_k = y$. By assumption, we know there is some v_i, v_j with $v_i = v_j$ but $i \neq j$. Assume $i < j$ then this requires a walk $v_{i+1} \ldots v_j$, however, the we know this is not the smallest so we can reduce it to $v_1 \ldots v_i v_{j+1} v_{j+2} \ldots v_k$. □

2 *Trees and Distance*

2.1 *Basic Properties*

In class, we observed the following properties for trees:

1. A tree $G = (V, E)$ has $|G| - 1$ edges.
2. Trees must be acyclic!
3. It has the minimum set of edges such that it is a connected component.
4. Leaves have degree of 1.

This means that the empty graph is not a tree because it is not connected!

A graph with no cycle is **acyclic**. A **forest** is an acyclic graph. A **leaf** is a vertex of degree 1. A **spanning subgraph** of G is a subgraph with vertex set $V(G)$. A **spanning tree** is a spanning subgraph that is a tree.

Theorem: Every tree with at least 2 vertices has at least two leaves.

Proof. Consider the longest path in the tree. The start of the path v_1 and the end of the path v_n can only connect to other vertices on the path. However, if we add another vertex connecting v_1 or v_n, it would create a cycle, which contradicts the definition of a tree. Therefore, start and end must be leaves. □

Theorem: If G is a connected graph, then G contains a spanning tree $T \subseteq G$.

Proof. Start with G keep removing while retaining the property of being connected to get a subgraph T which is minimally connected. □

Theorem: The following are equivalent

1. G is a tree.
2. G is connected and $e(G) = |G| - 1$.
3. G is minimally connected ($G - e$ is disconnected for all $e \in E(G)$).
4. G is maximally acyclic ($G + e$ has a cycle for all $e \notin E(G)$).

Proof. $1) \implies 2)$:
Assume that G is a tree, then by definition of a tree we get that G is connected. To prove the second part of 2), we have the claim:

Claim: Given a graph H and $e \notin E(H), e = xy, x, y \in V(H)$. Either x, y are in the connected component, or x, y are in different connected components. In the first case, the number of connected components in H is equal to the number of connected components in $H + e$. In the second case, the number of connected components of H is one more of $H + e$.

So now, we can build G one edge at a time, starting with $\overline{K_n}$ where $n = |G|$. Note that the starting number of connected components is n. If we add an edge and G_i is the current graph [2]. We cannot add an edge into one of G_i's connected components because then we would create a cycle because if an edge $e = xy$ then there is already a path from x tp y but $P_{xy} \cup xy$ is a cycle, contradiction!

[2] G_i denotes that i edges have been added.

Consequently, in the second case of the claim where the number of connected components drop by 1, therefore the number of edges added in is equal to the amount of connected components (which is $n - 1$). □

Proof. 2) $\implies$ 1)
Now, assume that G is connected and $e(G) = |G| - 1$. Remove edges from G one at a time while preserving connectivity. You will end up with a graph H that is connected, and acyclic, which by definition is a tree! As a consequence, we have

$$\begin{aligned} e(H) &= |H| - 1 \quad \text{This is because } H \text{ is a tree, so by definition.} &(17)\\ &= |G| - 1 \quad \text{We remove edges, but not vertices} &(18)\\ &= e(G) \quad \text{By assumption.} &(19) \end{aligned}$$

Since H and G has the same number of vertices, so G itself must be a tree too. □

Proof. 1) $\implies$ 3)
Assume that G is a tree. We know by definition of a tree that G is connected. We now want to show that it is *minimally* connected. Suppose that $e = xy \in E(G)$, we want to show that $G - e$ is connected. If not, note that in $G - e$ there would be a path P connecting x to y, but then $P \cup xy \in G$, which is a cycle. Contradiction! □

Proof. 3) $\implies$ 1)
Assume that G is minimally connected. Hence, by assumption we know G is connected. It remains to show that G is acyclic. If $c \in G$ is a cycle, the removing the edge of c keeps G connected, but this is a contradiction since we assumed removing any edge would remove the graph! □

Proof. 1) $\implies$ 4)
Suppose that G is a graph, it remains to show that G is maximally acyclic. Suppose $e = xy$ where $x, y \in V(G)$ and $xy \notin E(G)$. Now we show that $G + e$ contains a cycle. Indeed, if P is a path from x to y in G then $P + e$ is a cycle in $G + e$. □

Proof. 4) $\implies$ 1)
Assume that G is maximally acyclic, it remains to show that G is connected. Assume for contradiction that G is not connected. Let $x, y \in V$ be in different components. $G + xy$ has no cycle because if $c \in G$ were a cycle it would use xy and then $c - xy$ connects xy in G. This is a contradiction since x, y are in different components. □

3 *Hamiltonian Cycles and Eulerian Walks*

3.1 *Hamiltonian Cycles*

A **Hamiltonian graph** is a graph with a spanning cycle, also called a **Hamiltonian cycle**, e.g. if C is a subgraph of G, then $V(C) = V(G)$.

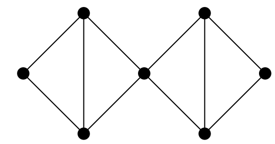


Figure 10. This graph does not contain a Hamiltonian cycle, as the middle node is repeated, and cycles by definition do not contain repeated vertices!

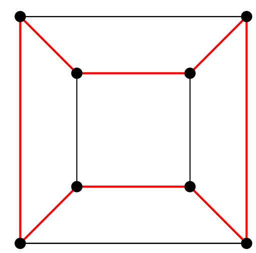


Figure 11. This contains multiple Hamiltonian cycles, note that you do not need to use all the edges, but you do need to use all vertices!

Theorem Bondy-Chvátal:
If G is a connected graph, $e = xy \notin E(G)$ and suppose $d_x + d_y \geq n = |G|$, then G has a Hamiltonian cycle if and only if $G + e$ has a Hamiltonian cycle.

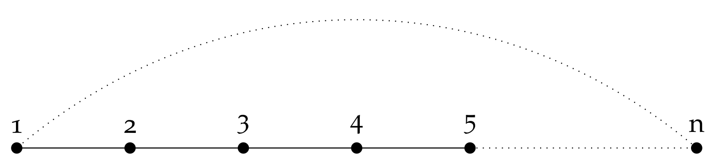
Corollary: If G is a graph where $|G| \geq 3$ such that $\forall x, y \in V(G)$ with $xy \notin E(G)$ we have $d_x + d_y \geq n$. Then G has a Hamiltonian cycle.

Proof. Corollary:
Repeately apply Bondy-Chvátal until you get a complete graph K_n. G being Hamiltonian $\iff$ K_n is Hamiltonian. □

Proof. Bondy-Chvátal:
We want to show that $G + e$ has a Hamiltonian cycle $\implies$ G has a Hamiltonian cycle (the other direction is trivial).

<u>Case 1</u>: If $c \subseteq G + e$ does not use e, then $c \subseteq G$.

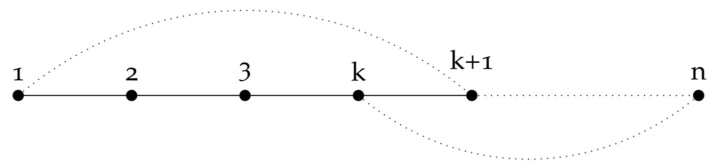
<u>Case 2</u>: List the vertices of c as $1, 2, \ldots, n$ such that the edges are $12, 23, \ldots, (n-1)n$ and $e = 1n$.



Now, look for a situation where the first vertex connects to $k+1$ and the last to k, formally we want to show

$$\exists 1 \leq k \leq n-1 \text{ s.t. } k+1 \in N(1) \land k \in N(n) \tag{20}$$

Visually, it looks like



Denote $N(1) = A$ and $N(n) = B$, we want $k \in B, k+1 \in A$. Also denote $A^{-1} = \{m : m+1 \in A\}$. We want a specific case when $A^{-1} \cap B = \varnothing$. Let $|A| = d_1$ and $|B| = d_2$, and $A \subseteq \{2, \ldots, n\}$, $B \subseteq \{1, \ldots, n-1\}$, $A^{-1} \subseteq \{1, \ldots, n-1\}$, so both B and A^{-1} are subsets of the set containing $n-1$ elements, and hence $|B| + |A^{-1}| \geq n$, so by the pigeonhole principle, we find the k. □

3.2 *Eulerian Circuit*

A **circuit** in a graph is a sequence of vertices $v_1 v_2 \ldots v_k$ possibly with repeats s.t. $v_i v_{v+1}$ is a distinct edge for all i and $v_k = v_1$ (cycles that allows repeat vertices, but not repeat edges).

Eulerian Circuit is a circuit using all edges of G.

A **trail** does not require $v_k = v_1$, it is a walk with no repeated edges. An **Eulerian Trail** is a trail uses all edges but does not require $v_k = v_1$. We say a graph G is **Eulerian** if G is an *Eulerian Circuit*.

G is Eulerian $\iff$ All vertices have even degree and G is connected.

Proof. $\Longleftarrow$

We proof by induction on the number of edges. Assume that all vertices have an even degree and G is connected. Then, we know there must be a circuit c, and with a circuit there must be a cycle too. If it did not have a cycle it would've been a tree since there exist leaf nodes with degree 1.

Now, consider the graph $G - e(c)$ (G without the edges of c). We would obtain

$$H = H_1 \sqcup \cdots \sqcup H_k \tag{21}$$

If we can find a Eulerian circuit in each of $H_1 \ldots H_k$, we can combine each using the original circuit, because we can follow c and when we enter H_i for the first time, augment c with H_i's Eulerian circuit.

We now want to show each H_i has all vertices of even degree, this is equivalent of showing $H_1 \sqcup H_2 \sqcup \cdots \sqcup H_k$ has all vertices even degree.

$$d_{x,H} = \underbrace{d_{x,G}}_{\text{even by hypothesis}} - \underbrace{d_{x,c}}_{\text{even by lec 1}} \tag{22}$$

Thus, $d_{x,H}$ is even too. $\square$

$\underbrace{\text{Eulerian Trail}}_{A} \iff$ At most 2 connected vertices have odd degree

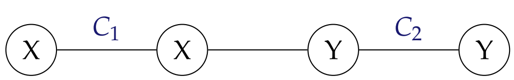
$\iff \underbrace{\text{Exactly 0 or 2 vertices have odd degree and is connected}}_{B}$.

Proof. **I dont understand this proof yet, visit OH!** $A \impliedby B$
Assume Exactly 0 or 2 vertices have odd degree and is connected.

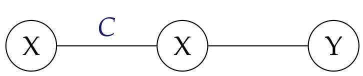
0 vertices with odd degree:
Then we trivially have an Eulerian circuit, which is also an Eulerian trail.

2 verticies $x, y \in V(G)$ with odd degree:
If $x\overline{y} \notin E(G)$, consider $G + e$ where $e = xy$, we can get an Eulerian Circuit then remove e.

If $xy \in E(G)$, $G - xy$ has all vertices of even degree. If $G - xy$ has 2 connected components C_1, C_2, we can get an Eulerian trail as



Other case, if $G - xy$ is connected



$\square$

4 Matchings

4.1 Bipartite Graphs

A **bipartition** of a graph G is a partition $V = X \sqcup Y$ such that X and Y are independent subsets of G. We say G is **bipartite** if there exists a bipartition.

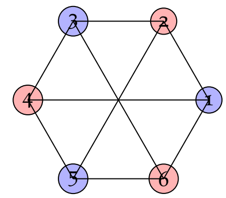


Figure 12. We see that there exists a bipartition for this graph as coloured.

We can tell if a graph is bipartite by identifying whether there are any odd cycles. If there are, then it means some edge will have the same colouring! Usually this is done by an 'algorithm' where we choose some vertex, and we place it in one of the two colours, which it will constraint its neighbours colours. Repeat until it terminates successfully or fails.

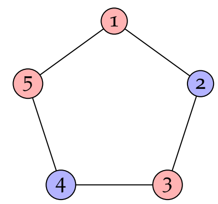


Figure 13. Notice that the vertices 1 and 5 have the same colour! It is impossible to colour this graph such that all adjacent vertices have different colours as it has an odd cycle.

We define $K_{m,n}$ to be a bipartite graph with m vertices in X and n vertices in Y where X, Y are bipartitions. Note that $K_{m,n} \cong K_{n,m}$. Every bipartite graph is a subset of these graphs, so how many possible edges are there? At most, there will be $m \times n$ edges, since we maximize m, n by doing a near perfect split. (E.g. we have 13 vertices, the near perfect split is $6 \times 7 = 42$ edges.)

A **proper bicolouring** of G is an assignment of b=blue or r=red to every vertex of G such that $bb, rr \notin E(G)$.

G biparitie $\iff$ G has a bicolouring.
biparition $\iff$ proper bicoloruing.

The following are equivalent:

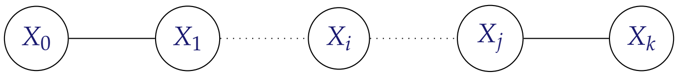
1. G is bipartite.
2. Every cycle has even length.
3. Ever circuit has even length.

Proof. $1 \implies 2$:
Assume that G is bipartite. If G had an odd cycle, then the odd cycle is not properly bicolourable. □

Proof. $1 \implies 3$:
Assume that G is bipartite. In a circuit, every 2nd vertex has a different colour, and cannot be odd for parity reasons (we assumed G is bipartite, so then every circuit has even length. □

Proof. 3 $\implies$ 2: Cycles are just special cases of circuits. □

Proof. 2 $\implies$ 3. Assume that every cylce has even length. For the sake of contradiction, assume that every circuit has odd length. Consider the smallest such odd circuit (m edges), since it is odd it cannot be a cycle, otherwise it would violate 2. Thus, there must be a repeated vertex (otherwise it would be a cycle, and it would not be odd length). Consider the circuit below, where $X_0 = X_k = v$



Assume the circuit between X_i and X_j is the smallest odd circuit, but it can't be odd length by assumption (we assumed the smallest odd circuit length m. So it has even length, but removing this subcircuit would have a smaller odd circuit. Contradiction! □

Proof. 3 $\implies$ 1
Assume that every circuit has even length. We solve this one connected component at a time. Let H be one of these components and pick a vertex v inside this component. Pick $w \in V(H)$ and colour each vw with b if every walk $v \to w$ has an even number of edges, r if there is an odd number of edges. **I dont understand this proof yet, visit OH!** □

Theorem: The bipartite handshaking lemma
If G is a bipartite graph then we have

$$e(G) := \sum_{x \in X} d_x = \sum_{y \in Y} d_y \tag{23}$$

Proof. Lets show WLOG that $e(G) = \sum_{x \in X} d_x$, note that each edge touches exactly one vertex in X, so split edges by which vertex in X they touch number touch $x \in X$ is d_x. □

4.2 *Matching Theory*

A **vertex cover** $C \subseteq V$ is a subset of vertices such that each edge touches at least one element of C. Note that a vertex cover does not need to be minimal.

We define the **size of the minimum vertex cover** as $\text{cov}(G)$, where $\text{cov}(G) \leq |C|, \forall C$ being a vertex cover. In the textbook, they use $\beta(G)$ to denote this.

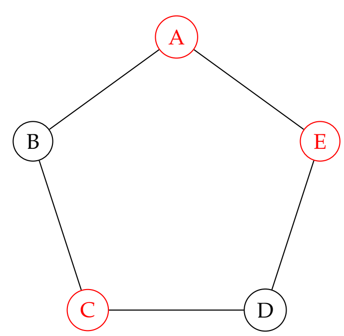


Figure 14. The vertices A, C, E is the minimal vertex cover, however, the vertices A, B, C, D, E is also a vertex cover, just not minimal.

The complement of a vertex cover is an independent set; vice-versa.

Proof. First, suppose that $C \subseteq V$ is a vertex cover. We want to show $V \backslash C$ is independent. If e is an edge in $V \backslash C$ then ☹ because e doesn't touch C.

Now, let $I \subseteq V$ be independent, we want to show $V \backslash I$ is a vertex cover. Indeed, if e is an edge of G, e cannot connect two vertices of I, since I is independent, so e touches at least one vertex from $V \backslash I$. Therefore, $V \backslash I$ is a vertex cover. □

The **size of the largest independent set** in G is defined as $\alpha(G)$.

Theorem: $\text{cov}(G) + \alpha(G) = |G|$

Proof. If C is the *minimal* vertex cover, then $V \backslash C$ must be the *largest* independent set. Consequently $|G| = |C| + |V \backslash C| = \text{cov}(G) + \alpha(G)$. □

A **matching** $M \subseteq E$ is a subset of edges such that no edges share a vertex. Note that $2|M| \leq |V|$.

$\text{match}(G) :=$ **size of the maximum matching**, in the textbook, they use $\alpha'(G)$ to denote this.

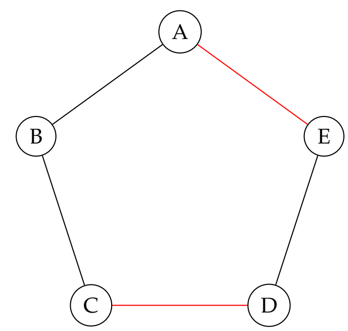


Figure 15. In red, a matching, in fact, this is the maximal matching. We could also have $\emptyset \subseteq E$ be a matching, but you cannot have a matching that is larger than 2 for this graph.

Theorem
$\text{match}(G) \leq \text{cov}(G)$

Proof. We will show for C vertex cover, M matching, that $|M| \leq C$. We find $f : M \to C$, which is injective $f(e_1) \neq f(e_2)$ if $e_1 \neq e_2$. To do this, for each edge e in M, let $f(e)$ be one of the two vertices which happens to lie in C. Because M is a matching, we cannot have $f(e_1) = f(e_2)$ for distinct edges in the matching because then this would be a common vertex of e_1 and e_2 which contradicts that M is a matching. [3] □

[3] Note to self, this proof wants to show the number of edges in M is at most the number of vertices in C, so we defined a function f that maps each edge in M to a vertex in C that it touches, because each edge in M has two endpoints, and at least one of these endpoints must lie in C as C is a vertex cover. Since M is a matching, meaning no two edges share a vertex, this function f has to be injective, meaning that different edges in M are mapped to different vertices in C.

Edge covering is a subset $F \subseteq E$ such that every vertex is contained in at least one edge of F. We have that $2|F| \geq |G|$.

$\text{Ecov}(G)$ is the minimum size of the edge covering. Requires G to have *no* isolated vertices. In the textbook, they use $\beta'(G)$ to denote this.

Theorem
If G has no isolated vertices, then $\text{match}(G) + \text{Ecov}(G) = |G|$

Proof. We first show that $\text{match}(G) + \text{Ecov}(G) \leq |G| \iff \text{Ecov}(G) \leq |G| - \text{match}(G)$. So we want to show $\exists$ some *edge covering* F such that $|F| \leq |G| - \text{match}(G)$ implies $\text{Ecov}(G) \leq |F| \leq |G| - \text{match}(G)$. Let M be the max matching. We want to show that there exists an edge covering F such that $|F| \leq |G| - |M|$. Take

$$F = M \cup \{\text{one edge touch each vertex not in the matching (repeats okay)}\} \tag{24}$$

where M accounts for $2|M|$ vertices in the graph, remaining edges in F has $|F| \leq G - 2|M|$, so we get

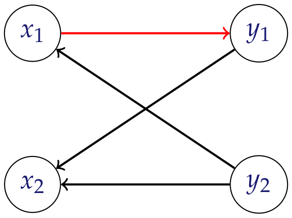
$$\begin{aligned} |F| &\leq |M| + |G| - 2|M| && (25) \\ &\leq |G| - |M| && (26) \end{aligned}$$

Now, we want to prove that $\text{match}(G) + \text{Ecov}(G) \geq |G|$. We want to show there exists a matching M such that $|M| \geq |G| - |\text{Ecov(G)}|$. Let F be an edge cover such that $|F| = \text{Ecov}(G)$. We want to show there exists a matching with $|M| \geq |G| - |F|$. Take the subgraph $H \subseteq G$ whose edges are F. We can start with an empty graph, and add one edge at a time, we end up with H, each time we add an edge the number of connected components can go down by at most 1. Thus, the number of connected components of $H \geq |G| - |F|$. Note that every connected component of H has at least one edge, otherwise it would be a single vertex, which contradicts the definition of an edge cover. Hence, we take M as one edge from each connected component. □

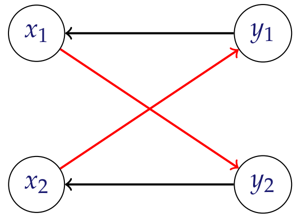
Theorem
For all graphs G, $\text{match}(G) \leq \text{cov}(G)$.

König's Theorem
For a biparitite graph G, $\text{match}(G) = \text{cov}(G)$.

Proof. Suffices to find a single match M and vertex cover C such that $|M| = |C|$. Take our bipartite graph, draw an arrow going *down* if the edge is in our matching, upwards if not. **We ask, can we reach an unmatched vertex in Y from an unmatched vertex in X?** If yes, we take a directed path which does this and flip alle dges from being in/out of matches,new matching M' is still a matching, with $|M'| = |M| + 1$.

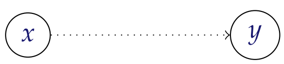


Initial Matching (1 edge)

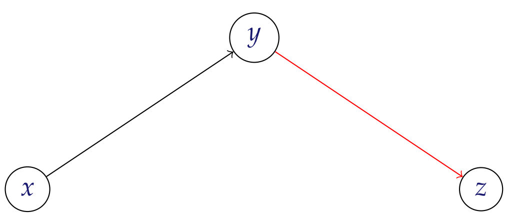


Inverted Matching (2 edges)

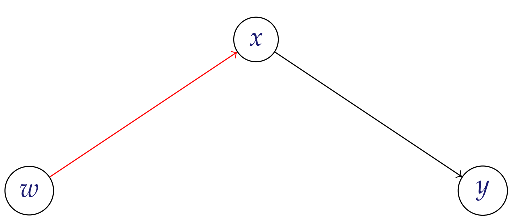
M will be the desired matching. Say that an edge of the matching is explorable if we can cross it eventually starting with unmatched vertex in X. Say that an edge of the matching is explorable if we can cross it eventually starting with unmatched vertex in X. C = tails of all explorable edges + tips of non-explorable. We want to show C is a vertex cover. Take edge $e \in E$, want to show e touches C. If $e \in M$, then by construction it is in C as we take every tip or tail of an edge that is in the matching. If $e \notin M$, we ask *can I get to this vertex $y \in Y$ from an unmatched vertex $x \in X$?*.



Consider the case if the answer is **YES**. Then y is in edge of matching that our unmatched vertex x has an edge with, and so it is covered. In this case, we have something as illustrated below, where $yz \in M$, so yz is explorable, so $y \in C$.



Consider the case if the answer is **NO**. Then x our unmatched vertex that we start from, is also not reachable from an unmatched vertex X, so there is an edge $wx \in M$.



Thus, $x \in C$. □

G is a biparitite graph with bipartition $X \sqcup Y$, an X-**perfect matching** is a matching that touches all vertices in X. In X-perfect matching, if $A \subseteq X$, then $|\Gamma(A)| \geq |A|$.

Halls Theorem
If $\forall A \subseteq X$, $|\Gamma(A)| \geq |A|$ then $\exists$ X-perfect matching.

Halls Marriage Theorem
A graph with bipartition $X \sqcup Y$ has an X-perfect matching *if and only if* $\forall A \subseteq X$ we have $|\Gamma(A)| \geq |A|$.

Proof. $\Longrightarrow$: Trivial because X-perfect matching guarantees $\Gamma(A) \geq |A|, \forall A \subseteq X$ because $\Gamma(A)$ contains the part of opposite endpoints of the matching which touches A.

$\Longleftarrow$: We now want to show $\exists$ matching M where $|M| = |X|$, equivalently, $\text{match}(G) = |X|$. By Königs theorem, this is equivalent to showing $\text{cov}(G) = |X|$. If we took $C = X$, this would be a vertex cover, therefore $\text{cov}(G) \leq |X|$. It remains to show $|C| \geq |X|$.

$$\begin{aligned}
|C| &= |C \cap X| + |C \cap Y| && (27)\\
&\geq |C \cap X| + |\Gamma(X \backslash C)| && (28)\\
&\geq |C \cap X| + |X \backslash C| && (29)\\
&= X && (30)
\end{aligned}$$

where we used the hypothesis that $X \backslash C \subseteq X$, so $|\Gamma(X \backslash C)| \geq |X \backslash C|$. $\square$

We say a matching M is X-perfect of defect d if $|M| = |X| - d$.

Defect Hall's Marriage Theorem: A defect d X-perfect matching exists *if and only if* $\forall A \subseteq X$, $|\Gamma(A)| \geq |A| - d$.

Proof. $\Longrightarrow$: For a bipartite graph we add d new vertices to Y. Fully connect X with the added vertices and call this graph G'. Then,

$$\begin{aligned}
|\Gamma_{G'}(A)| &= |\Gamma_G(A) \cup \{d\}| && (31)\\
&\geq |A| - d + d && (32)\\
&\geq |A| && (33)
\end{aligned}$$

Now apply Hall's theorem in G' and an X-perfect matching would involved the extra vertices. Deleting those extra vertices defect the matching by at most d.

$\Longleftarrow$: Given finite sets $S_1, \dots, S_k$, a **transversal** is a choice of elements $x_i \in S_i, \forall 1 \leq i \leq k$ such that all x_i are distinct. For example:

$$\{1, 2, 5\}, \{3, 4\}, \{4, 5\}, \{5\}$$

Then, there exists a transversal if and only if $\forall A \subseteq \{1, \dots, k\}, \bigcup_{i \in A} S_i \geq |A|$. We construct bipartite graph, $X = \{S_i\}, Y = \{e \in S_i\}$. Neigbourbood size is the size of the union. $\square$

5 Max Flow Min Cut

A **directed graph** is $G = (V, \overrightarrow{E})$ where $\overrightarrow{E} \subseteq V \times V$ is the ordered pairs with $(v, v) \notin \overrightarrow{E}$ and $(v, w) \in \overrightarrow{E} \implies wv \notin \overrightarrow{E}$.

A **flow network** is a directed graph G together with distinct vertices $s, t \in G$ such that $\overrightarrow{vs} \notin \overrightarrow{E}, \overrightarrow{tw} \notin \overrightarrow{E}, \forall v, w \in V$.

A **capacity** on a flow network is a function $c : \overrightarrow{E} \to \mathbb{R}_{\geq 0}$.

A **flow** within a flow network is an assignment $f : \overrightarrow{E} \to \mathbb{R}_{\geq 0}$ subject to the following two constraints:

1. $\forall \overrightarrow{E} \in \overrightarrow{E}, f(\overrightarrow{e}) \leq c(\overrightarrow{e})$.
2. $\forall v \in V \backslash \{s, t\}$ we have $\text{flow}_{\text{in}}(v) = \text{flow}_{out}(v)$ where $\text{flow}_{\text{in}}(v) = \sum_{\overrightarrow{xv} \in \overrightarrow{E}} f(\overrightarrow{xv})$.

Theorem
$\text{flow}_{\text{out}}(s) = \text{flow}_{\text{in}}(t)$

Proof. $\forall v \in V \backslash \{s, t\}$, we have $\text{flow}_{\text{in}}(v) - \text{flow}_{\text{out}}(v) = 0$.

$$\sum_{v \in V \backslash \{s,t\}} \text{flow}_{\text{in}}(v) - \text{flow}_{\text{out}}(v) = 0 \tag{34}$$

In equation 34, if $\overrightarrow{e} = \overrightarrow{xy}$, we see $f(\overrightarrow{e}) - f(\overrightarrow{e}) = 0$. If $x = s, y \in V \backslash \{s, t\}$, we see only a $f(\overrightarrow{e})$. If $x \in V \backslash \{s, t\}, y = t$ we see $-f(e)$. If $x = s, y = t$ we see 0. Therefore, equation 34 is equivalent to

$$0 = \sum_{\overrightarrow{sy} \in \overrightarrow{E}} f(\overrightarrow{sy}) - \sum_{\overrightarrow{xt} \in \overrightarrow{E}} f(\overrightarrow{xt}) \tag{35}$$

$$= \text{flow}_{\text{out}}(s) - \text{flow}_{\text{in}}(t) \tag{36}$$

□

A **cut** is a collection of vertices containing s but not t. The value of a cut is defined as

$$\text{val}(W) = \sum_{\substack{\overrightarrow{xy} \in \overrightarrow{E} \\ x \in W, y \notin W}} c(\overrightarrow{xy}) = \text{cap}_{\text{out}}(w) \tag{37}$$

Theorem
val(flow) ≤ val(cut)

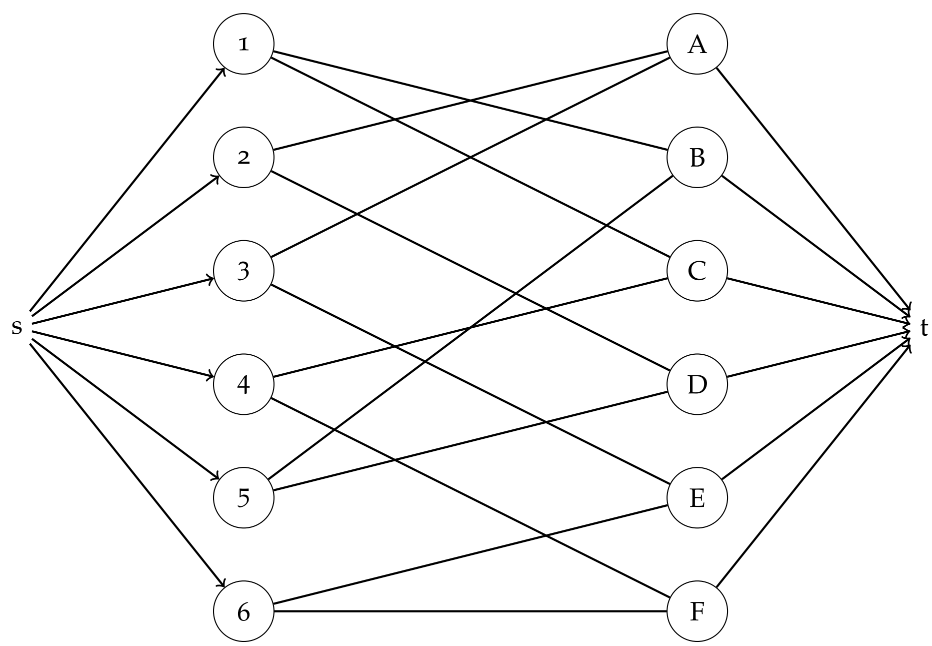
Proof. If we add the conservation equations for all $v \in W \backslash \{s\}$ then we get

$$\begin{aligned} \text{val(flow)} &= \text{flow}_{\text{out}}(w) - \text{flow}_{\text{in}}(w) && (38)\\ &\leq \text{cap}_{\text{out}}(w) - 0 && (39)\\ &= \text{val}(\text{cut}) && (40) \end{aligned}$$

□

Theorem: Maximum Flow = Minimum Cut, use Ford-Fulkerson to find Maximum Flow, and then you can find a minimum cut and see values are the same. Proofs done in CSC373H1.

Consider a bipartite graph that has a sink t connected to all Y and a source s connected. We need a capacity so let us assign 1 to all the edges.



Not exactly identical to the one drawn in lecture!

What can we say about max flow / min cut? We notice that in attempting to create a max flow, we create a matching. This implies that a maximum flow and a maximum matching are equivalent. In the lecture, we defined

$$L = \{1,2,3\} \quad Y' = \{4,5,6\} \quad K = \{A,B,C\} \quad X' = \{D,E,F\}$$

And so, the value of the cut is

$$\mathrm{val}(c) = |L| + (\#\text{ of edges } K \to Y \backslash L) + |X \backslash K| \tag{41}$$

For König, want to produce vertex cover where size is $\leq$ size of matching max flow min cut says max matching is

$$|L| + |X \backslash K| + (\#\text{ of edges } \ \to Y \backslash L) \tag{42}$$

for some cut $\{s\} \cup K \cup L$.

Suppose we change the capacities to say, ∞ for edges between X, Y. Then it does not matter given that there is a contstraint given that each edge with ∞ capacity is flowed both into and out of with capacity 1. But the value of the cut is

$$\mathrm{val}(c) = |L| + \infty \times (\#\text{ of edges } K \to Y \backslash L) + |X \backslash K| \tag{43}$$

In a minimum cut, there are no edges $K \to Y \backslash L$. The maximum matching is $|L| + |X \backslash K|$.

6 Connectivity

6.1 Vertex Connectivity

Moral Definition
A graph G, the vertex connectivity $\kappa(G)$ is the number of vertices needed to disconnect the graph. *Unless* $G = K_n$ *and then* $\kappa(K_n) = n-1$.

Alternate Definition
$\kappa(G)$ is the smallest size of a *vertex-separator* $S \subseteq V$ such that $G - S$ is either a single point or disconnected.

$\kappa(G) \leq \delta(G)$ where $\delta(G)$ is the minimum degree of any vertex in G.

Proof. Let v have the smallest degree d. Take $S = \gamma(v)$. We want to show S is a vertex separator. Consider $G - S$, v is not connected to any vertex in $G - S$.

- Case 1: $G - S$ contains more vertices than v because v is not linked to the rest of the graph.
- Case 2: $G - S$ is a single vertex v.

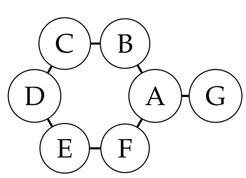
In either case, S is indeed a vertex separator. □

A graph is k-**connected** if $\kappa(G) \geq K$.

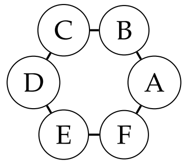
- 0-connected: All graphs except $\varnothing$
- 1-connected: All connected graphs except K_1.
- 2-connected: Graphs with no cut vertex.

Remark: G is k-connected $\iff$ $|G| \geq K+1$ and no set $S \subseteq V$ with $|S| \leq k-1$ disconnects G.

Warning



Removing vertex G increases vertex connectivity, this graph has $\kappa(G) = 1$



This graph, despite having fewer vertices, has $\kappa(G) = 2$

Let e be an edge of G. We have

$$\kappa(G) - 1 \overset{*}{\leq} k(G-e) \overset{**}{\leq} \kappa(G) \tag{44}$$

Proof. Start with $**$. Consider a minimal vertex-separator S for G. If we can show , S is a vertex separator for $G-e$ then $\kappa(G-e) \leq |S| = \kappa(G)$. We now want to show $G-e-S$ is disconnected or a single point.

- $G-S$ is a single point, then $G-e-S$ is a single point also.
- $G-S$ is disconnected, then if e is incident to $G-e-S = G-S$. Then we are done. Otherwise, if e is not then $G-e-S = (G-s)-e$, disconnected because $G-s$ is.

Now we show $*$. Suppose S is a vertex separator for $G-e$, want to show there exists a vertex separator for G with at most $|S|+1$ vertices.

- $G-e-S$ single point then $G-S$ is a single point and $G-e$ is a single point, so S is a vertex-separator for G.
- $(G-e)-S$ is disconnected, then if $G-S$ is disconnected as well we are done. In the case $G-S$ is connected, consider $G-S$ as two connected components H_1, H_2 connected by edge e. If H_1 has at least two vertices, then $G-S-x$ is disconnected for $x \in H_1$, so $S \cup \{x\}$. If $|H_2| \geq 2$ then $G-S-y$ disconnected for $y \in H_2$, so $S \cup \{y\}$. If $G-S = xy$, then $G-S-x$ is a single vertex, so $S \cup \{x\}$.

$\square$

G connected, s, t distinct vertices in G *and* $st \notin E(G)$, we define a s, t vertex-separator $S \subseteq V \backslash \{s, t\}$ such that in $G-S$, s, t are not linked. The minimum size of s, t vertex separator is $\geq$ largerst number of **vertex disjoint paths** (excluding s, t) s to t.

Vertex Mengers Theorem:
The minimum size of s, t vertex-separator is equal to the largest number of vertex disjoint paths s to t.

Proof. I did not understand this... not testable though! $\square$

Vertex Menger's Corollary
G is k-connected $\iff$ for every 2 distinct vertices s, t there are at least k-vertex (excluding s, t) disjoint paths from s to t.

Proof. $\implies$. Assume that the graph G is k-connected. Being k-connected means $|G| \geq k+1$ and no subset $S \subseteq G$ with $|S| \leq k-1$ has $G-S$ disconnected.

Case 1: $st \notin E(G)$ then every s, t vertex-separator is of size greater than k.

Case 2: Consider $st \in E(G)$ Consider $G-e$. $\kappa(G-e) \geq \kappa(G)-1 \geq k-1$. If we apply Vertex Menger's once more, we get that in $G-e$ we have $k-1$ vertex disjoint paths. Together with e, this is k many paths.

$\impliedby$ First show $|G| \geq k+1$. Take $s \neq t$ in G and consider the k-paths. Each path contributes at least one new vertex to G (other than a direct $s-t$ path). $|G| \geq 2+(k-1) = k+1$. Now we want to show if $S \subseteq V$ with $|S| \leq k-1$ then $G-S$ is connected. Suppose otherwise, i.e. $G-S$ is disconnected. Let s, t be in 2 different connected components of $G-S$. Then, S is a s, t vertex separator, which can happen since there are k vertex disjoint paths $s \to t$. □

6.2 *Edge Connectivity*

G a graph, the **edge-connectivity** $\lambda(G)$ is the smallest number of edges needed to be removed so that G becomes disconnected (or a single vertex).

A **bridge** is an edge in a connected graph whose removal disconnects the graph.

For example, when $\lambda(G) = 0$, then it is already disconnected, an example of $\lambda(G) = 1$ is for example, a bowtie. We want to relate number of edges needed to be cut to separate s and t to the number of edge-disjoint paths $s \to t$ like with vertex connectivity. Consider the bowtie, there is a single vertex-disjoint paths. However, there are actually two edge disjoint paths.

The **line graph** $L(G)$ of G is the graph whose vertices are the edges of G, with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G.

Edge Menger's Theorem
If x and y are distinct vertices of a graph G, then the minimum size of an x, y-disconnecting set of edges equals to the maximum number of pairwise edge-disjoint x, y-paths.

G is k-edge connected if $\lambda(G) \geq k$.

Edge Menger's Corollary
G is k-edge connected *if and only if* every pair of vertices s, t is connected via k edge disjoint paths.

7 Planar Graphs and Coloring

The question to ask is *can I color the map with 4 colors such that no two regions sharing an edge uses the same color*.

A (proper) r-**coloring** of a graph G is an assignment $X : V \to \{1, \ldots, r\}$ so that if vw is an edge then $\chi(v) \neq \chi(w)$.

A **planer graph** is a graph $\mathcal{G}$ that can be using polygonal (zig-zag) edges, so that no two edges cross or touch.

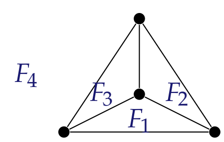
A **plane** graph is a planar graph + its drawing.

The easiest way to show planar could draw a plane graph.

The **face** of a plane graph denoted as F is a *connected component* of $\mathbb{R}^2 - G$ (i.e. remove all edges + vertices)

$x, y \in \mathbb{R}^2 - G$ are **linked by a polygonal arc** if we can draw zig-zag avoiding G.

The **closure** is $\overline{F}$ is defined as $\overline{F} = F + \text{ limit points in } G$.



The above graph has 4 faces, three bounded inner faces and one unbounded outer face.

The **boundary** of a face F, denoted as ∂F, is the set of edges and vertices that enclose F. It forms a closed walk along the edges of the plane graph G.

Theorem
Every plane drawing of a forest has 1 face.

Proof. We proof by induction on the number of edges.
Base Case: If there are 0 edges then it is trivial, it consists only of isolated vertices. A plane drawing of such a forest has exactly one face, the entire unbounded region of the plane.

Step Case: Assume that any plane drawing of a forest with k edges has exactly one edge. Take a plane drawing forest F_1 with $k+1$ edges and take a leaf edge e_{xy} with y having degree 1. By induction, $F-e$ has 1 face. Now add the edge back in. Basically say we can have an algorithm to eliminate crossings and so we would not have a zigzag, can take facts like these for granted in this course. □

Corollary
I dont quite understand this proof and I think the corollary is problematic, visit office hours. If G is *not* a forest, then every face boundary contains a cyle.

Proof. Take a face F. If ∂F does not contain a cycle then it is a forest $H \subseteq G$. Every two points in F are polygonal path in H, but also even two points in $\mathbb{R}^2 - H$ are connected by path so $\mathbb{R}^2 - H = F$ so $G = H$. □

Theorem
Every drawing of a cycle has an "inside" and "outside" (the drawing has two faces).

Proof. Choose a random direction not parallel to any edge. For any $p \in \mathbb{R}^2 - C$ consider a ray $p + \mathbb{R}_{\geq 0}$. □

TODO: FINISH THE NOTES ON THIS SECTION!!!

7.1 *Colouring*

A **proper** r**-colouring** of a graph G is a function $C : V \to \{1, \dots, r\}$ such that $C(x) \neq C(y)$ when $xy \in E$.

The **chromatic number** of a graph G, $\chi(G)$ is the smallest r such that G has a proper r-colouring.

A k**-partite graph** G is a graph such that we can partitoin $V = V_1 \sqcup \cdots \sqcup V_k$ such that $G[V_i]$ has no edges. V_i could be empty.

Theorem: $\chi(G) \leq k \iff G$ is k-partite.

$\omega(G)$ is the largest r such that $K_r \subseteq G$, aka the **clique number**, which is the maximum size of a clique.

$\chi(G) \geq \omega(G)$

$\chi(G) \leq \Delta + 1$ where $\Delta = \max_{v \in V} d_v$.

Proof. Colour the vertices of G one at a time, using one of $\Delta + 1$ colours. You never get stuck because if v is the current uncoloured vertex, then its neighbours have been coloured with at most $|\Gamma(v)|$ many colours, which is less than $\Delta(G) + 1$ (the number of colours available). □

A graph G is k-degenerate if we can list the vertices $v_1, \ldots, v_n$ such that v_i has degree $d_{v_i} \leq k$ in $G \backslash \{v_1, \ldots, v_{i-1}\}$.

If a graph G is k-degenerate then $\chi(G) \leq k + 1$

Proof. Greedy algorithm in the order $v_n, \ldots, v_i$. There are only k edges going forward, so worst case we use $k + 1$ colours. □

The 6-colour theorem states, every plane or graph has chromatic number $\chi(G) \leq 6$.

G is planar $\implies$ G is 5-degenerate.

We first prove a claim that *any planar graph has vertex of degree* ≤ 5.

Proof. Choose any connected component of G. If every vertex has degree ≥ 6, then then number of edges $\geq \frac{1}{2}6|H| \geq 3|H| > 3|H| - 6$ (planar graphs H cannot have more than $3|H| - 6$ edges. □

Proof. $G - v_1$ is planar means v_2 is a vertex of degree less than 5 in $G - v_1$. $G - v_1 - v_2$ is planar means v_3 is a vertex of degree less than 5 in $G - v_1 - v_2$. □

5-colour theorem
Every planar graph has $\chi(G) \leq 5$.

Proof. We prove by induction on G.
Base Case: Empty graph, trivial.
Step Case: Choose a vertex v such that $d_v \leq 5$ (since it is a planar graph, we know such exists), colour $G - v$ with 5 colours (by induction we can do this). Consider the following subcases:

- If $d_v \leq 4$ and v was the only vertex not coloured, we have 5 colours, and pick the available colour.

- If $d_v = 5$ and there is a repeated colour in $\Gamma(v)$, then there is a colour available.
- If $d_v = 5$ but all its neighbors are coloured with 5 different colours, we are going to recolour G using **kempe-chains**[4], such that there is a duplicate colour around v_1. Denote the colours as C_1, C_2, C_3, C_4, C_5, and consider the C_3-C_1 kempe chain containing v_1. If it does not contain v_3 (v_3 having C_3) we can invert its colour. Swap colours and colour v with C_1. [5].
  - If however, the C_3-C_1 kempe chain seperates v_2 from v_4 and v_5, consider the C_2-C_4 kempe chain containing v_2. It does not contain v_4 as the previous kempe-chain seperates v_4 and v_2 so recolour v.

□

[4] If we can swap the colours of a kempe chain, the graph remains properly coloured

[5] Invert the colour of the kempe chain, and so C_1 becomes 'free' and we can colour v with C_1, and since the kempe chain does not contain v_3, we free up a colour surrounding v

A proper **edge colouring** of G is an assignment of colours to edges such that two edges sharing the same vertex have different colours. Colour classes are matching. $\chi'(G)$ is the smallest number of colours for edge colouring called **edge chromatic number**.

Theorem (Vizing)
$\Delta \leq \chi(c(G)) \leq \Delta + 1$

Proof. **TODO: FILL IN PROOF** □

8 Chromatic Polynomial

$P_G(r)$ is the number of proper r-colourings of G.

For example,

- $P_{\overline{K_n}}(r) = r^n$
- $P_{K_n}(r) = r(r-1)\dots(r-n-1) = \frac{r!}{(r-n)!}$
- $P_{P_n}(r) = r(r-1)^{n-1}$

Theorem
$P_G(r)$ is a polynomial for all graph G

For $e = xy \in E(G)$, G/e is the contraction (collapses the edge, the two vertices connected by e become the same vertex).

Theorem
$P_G(r) = P_{G\backslash e}(r) - P_{G/e}(r)$

Proof. $P_{G\backslash e}(r)$ counts the number of r-colorings of G which are proper except possibly on this one edge. $P_{G\backslash e}(r) = P_G(r)+$ the number of r-colourings of G which are proper to at all edges other than $e = xy$ and which give x, y the same colour. Now note that there is a proper bijection between proper r-colourings and ... sorry I did not understand the last part of this proof.

Alternative: $P_{G\backslash e}(r) - P_G(r)$ = colourings of G proper everywhere except $e = xy$ so $C(x) = C(y)$. □

Corollary
$P_G(r)$ is a polynomial in r, valid $\forall r \geq 0$ ($P_G(0) = 0$ if $G = \varnothing$, 1 if $G = \varnothing$.

Proof. We can prove by induction by inducting on the number of edges $e(G)$, simple as base case is just $P_{\overline{K_m}}(r) = r^m$, and we assume true for all smaller graphs (strong induction), so $P_G(r) = P_{G\backslash e}(r) - P_{G/e}(r)$ also holds since we can apply I.H. on $P_{G\backslash e}(r) - P_{G/e}(r)$. □

Theorem
$P_G(r) = r^{|G|}+$ lower order terms. Lower order terms like $r^m, m < |G|$

Proof. We induct on the number of edges $e(G)$, the base cases is easy as

$$P_{\overline{K_m}}(r) = r^m \tag{45}$$

I.H. assumes theorem holds for all smaller graphs, so we have

$$P_G = \underbrace{P_{G\setminus e}}_{r^{|G\setminus e|} + \text{lower order}} - \underbrace{P_{G/e}}_{r^{|G/e|} + \text{lower order}} = r^{|G|} + \text{lower order terms} \tag{46}$$

we applied the I.H. since $|G/e|$ and $|G\setminus e|$ are less than $|G|$. □

Remark
If $|G| = n$, then

$$P_{K_n}(r) \leq P_G(r) \leq P_{\overline{K_n}}(r) \tag{47}$$

An **acyclic orientation** of a graph G is a choice of direction for each edge so that no directed cycles occur.

We define A_G as the number of acyclic orientations of G.

Theorem

$$A_G = (-1)^{|G|} P_G(-1) \tag{48}$$

For instance, consider a triangle. We get eight total orientations, but two of them are cycles, so we have six good ones. Then, $P_{C_3}(r) = r(r-1)(r-2)$ and hence we get 6 if we substitute $r = -1$.

Look at acyclic orientations of $G\setminus e$, G, G/e, $e = xy$, split them up according to there is a path $x \to y$, $y \to x$, or neither.

	$A_{G\setminus e}$	A_G	$A_{G/e}$
$x \to y$	1	1	0
$y \to x$	1	1	0
Neither	1	2	1

TODO, draw the example graph for the table.

Proof. □

9 Ramsey Number

We define the Ramsey Number $R(a, b)$ to be the smallest number n so that every red-blue edge-Coloured K_n contains either a red K_a or a blue K_b.

We define $R(a)$ to be $R(a, a)$, where is the smallest n such that a red-blue edge-colored K_n contains a monochromatic K_n

Theorem The Ramsey number exists for all a, b and $R(a, b) \leq 2^{a+b}$

Proof. Induct on $a + b$.

Base Case: $a + b = 2$, such that $a = 1, b = 1$. $R(1, 1) = R(1) = 2 \leq 2^{a+b}$. Since having K_2 will make sure that either blue or red K_1 appears.

Inductive Step: Assume statement holds for $a + b \leq k$. Then for $a + b = k$, to show that $R(a, b)$ exists, we need to show the claim

Claim:

$$R(a, b) \leq R(a - 1, b) + R(a, b - 1)$$

To show the claim, let $n = R(a - 1, b) + R(a, b - 1)$, then every red-blue edge-colored K_n will should contains a red K_a or a blue K_b.

Pick an arbitrary vertex $v \in V(K_n)$, let $G' = K_n \setminus \{v\} = R \sqcup B$ to be the partition of the neighbours of v according to whether the edge connecting to v is red or blue, for example R will be the set of vertices such that it connected v as red in K_n.

Note that it must be the case that either $|R| \geq R(a - 1, b)$ or $|B| \geq R(a, b - 1)$. Prove by contradiction, assume both fails, we will have

$$\begin{aligned} |R| + |B| &\leq R(a - 1, b) - 1 + R(a, b - 1) - 1 \\ &= n - 2 \\ &< n - 1 \end{aligned}$$

This contradict to the fact that $|V(R \sqcup B)| = |V(G')| = n - 1$.

If $|R| \geq R(a - 1, b)$, then R will either contains a red K_{a-1} or a blue K_b. If K_b then done. If a red K_{a-1}, then adding v back to $|R|$, based on the construction definition of R. We know that v have red edge connected to all vertices in R as red where we could have K_a.

The proof for $|B| \geq R(a, b - 1)$ will be symmetric. Hence omit here. □

Example Application for Ramsey Theory: Suppose we have a sequence $x_1, \cdots, x_n$ of $n = R(10)$ distinct real number. Then there is either a increasing sequence of length 10 or a decreasing sequence of length 10.

Proof. Consider a graph K_n with vertices $\{1, \cdots, n\}$ given by. For edge jk where $j < k$, color the edge as

$$\begin{cases} \text{Increasing, if } x_j < x_k \\ \text{Decreasing, if } x_j > x_k \end{cases}$$

By Ramsey theory, there is either an Increasing K_10 or a Decreasing K_10. Where based on the construction, in the sequence, it will be a Increasing/Decreasing sequence with length 10. □

Define $R(a_1, a_2, \cdots, a_k)$ to be the smallest n such that if K_n is edge-colored with colors $c_1, \cdots, c_k$ then there must always be a $c_i$colored K_{a_i} for some i.

$R(a_1, \cdots, a_k)$ exists

Proof. Induct on k.

Base Case: $k = 2$. Proved in the first theorem.

Inductive Step: Assume this is right for all smaller k. Then let $n = R(a_1, R(a_2, \cdots, a_k))$, by Ramsey's theorem, there is either a $c_1$colored K_{a_1} or a $K_{R(a_2, \cdots, a_k)}$ where none of the edges are colored as c_1.

For the latter case, this means we have a $K_{R(a_2, \cdots, a_k)}$ edge-colored with $c_2, \cdots, c_k$. By induction hypothesis, there is a c_i colored K_{a_i} for some $2 \leq i \leq k$ □

Let G be the complete graph with infinite vertex set $\{1, 2, \cdots\}$ and edge set $\{(i, j) | i < j\}$ and suppose we have a red-blue edge-coloring of G. Then G contains a complete infinite monochromatic subgraph

Proof. Let $v_1 = \{1\}$, then we are going to do the following process:

1. Look at all edges from v_1 to other vertices $\{(1, j) | j \geq 2\}$
2. Among these edges, there must be either infinitely many red edges or infinitely many blue edges. (Based on the Pigeonhole Principle)
3. Name a color c_1, to color all these edges where incident to v_1

4. Remove all vertices that are colored as c_1

Then we let v_2 = the smallest non-deleted vertex. Do the above process again with v_2 and remove all vertices that are colored as c_2.

After infinitely steps, you will obtained a sequence $v_1, v_2, \cdots$ such that for each v_i, its forward neighbours have edges of the same color. $\square$

Example about infinite sequence Given an infinite sequence of distinct numbers $x_1, x_2, \cdots$ there is an infinite subsequence which is increasing or an infinite subsequence which is decreasing.

Proof. Apply the infinite Ramsay theorem to the coloring on $\{1, 2, \cdots\}$ where we colors ij with $i < j$ as "increasing" if $x_i < x_j$ and "decreasing" with $x_i > x_j$ $\square$

Theorem: For $a \geq 3$ we have $\sqrt{2}^a < R(a) \leq 4^a$

Proof. Since $R(a) = R(a, a) \leq 2^{a+a} = 2^{2a} = 4a$, Hence we already prove the upper bound.

Then we just need to show the lower bound. For $n = \sqrt{2}^a$, we want to show that there is a red-blue edge-colored K_n with no monochromatic K_a.

Select a random coloring. Since each edge is colored red or blue with equal probability $\frac{1}{2}$. Then the probability that any individual K_a

$$P(K_a \text{ is monochromatic}) = 2 \cdot \frac{1}{2}^{\binom{a}{2}} = 2^{1-\binom{a}{2}}$$

Then for n vertices, there are $\binom{n}{a}$ possible K_a.

$$P(\text{At least one } K_a \text{ is monochromatic}) = \binom{n}{a} \cdot 2^{1-\binom{a}{2}}$$

Then we simplify the bound.

- Approximate $\binom{n}{a}$

$$\binom{n}{a} = \frac{n!}{a!(n-a)!} \approx \frac{n^a}{a!}$$

Then plug in the approximation and $n = \sqrt{2}^a$, note that $n^a = (\sqrt{2}^a)^a = 2^{\frac{a^2}{2}}$. Then the probability becomes:

$$2^{\frac{a^2}{2}} \cdot 2^{1-\binom{a}{2}} = \frac{2^{\frac{a^2}{2+1}-\binom{a}{2}}}{a!}$$

For $\frac{a^2}{2} - \binom{a}{2}$ we can simplify as: $\frac{a^2}{2} - \frac{a^2}{2} - \frac{a}{2} = \frac{a}{2}$.

Hence the probability becomes

$$\frac{2^{\frac{a}{2}+1}}{a!}$$

Since this probability is strictly less than 1, then the probability that at least one K_a is monochromatic is less than 1, which means there exists a coloring where no monochromatic K_a exists. $\square$

A $k-$uniform hypergraph $\mathcal{G}$ is a finite set V called "vertices" and E a family of size K subsets of V called "hyperedges".

We define the complete $k-$uniform hypergraph $K_n^{(k)}$ to have vertex set V with $|V| = n$ and all possible $k-$element subsets as edge set.

Note if $k = 2$, then it is the definition of normal graph.

Example for $k = 3$ an example of $k-$uniform hypergraph is with $V = \{1, 2, 3, 4, 5\}$ and $E = \{123, 134, 234, 145, 135, 345\}$

We say that a $k-$uniform hypergraph G contains $K_a^{(k)}$ if there is a subset $A \subset V$ with $|A| = a$ such that every $k-$tuple in A lies in E.

Example In the previous example G contains $K_4^{(3)}$ on vertices $\{1, 3, 4, 5\}$ since E contains all $134, 135, 145, 345$

There exists a number $n = R^k(a, b)$ which we call the $k-$uniform Ramsey number, such that if hte edges of a complete $K_n^{(k)}$ are colored red or blue, then either is contains a red $K_a^{(k)}$ or a blue $K_b^{(k)}$.